THE TWISTED GRASSMANN GRAPH IS THE BLOCK GRAPH OF A DESIGN

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Abstract. In this note, we show that the twisted Grassmann graph constructed by van Dam and Koolen is the block graph of the design constructed by Jungnickel and Tonchev. We also show that the full automorphism group of the design is isomorphic to the full automorphism group of the twisted Grassmann graph.

1. Introduction

Let V be a $(2e+1)$-dimensional vector space over $\mathrm{GF}(q)$. If W is a subset of V closed under multiplication by the elements of $\mathrm{GF}(q)$, then we denote by $[W]$ the set of 1-dimensional subspaces (projective points) contained in W. We also denote by $\left[{W \atop k}\right]$ the set of k-dimensional subspaces of W, when W is a vector space. The geometric design $\mathrm{PG}_e(2e,q)$ has $[V]$ as the set of points, and $\{[W] \mid W \in \left[{V \atop e+1}\right]\}$ as the set of blocks. The block graph of this design, where two blocks $[W_1],[W_2]$ are adjacent whenever $\dim W_1 \cap W_2 = e$, is the Grassmann graph $J_q(2e+1,e+1)$ which is isomorphic to the Grassmann graph $J_q(2e+1,e)$.

For each prime power q and an integer $e \geq 2$, the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ discovered by van Dam and Koolen is a distance-regular graph with the same parameters as the Grassmann graph $J_q(2e+1,e)$. The twisted Grassmann graphs were the first family of non-vertex-transitive distance-regular graphs with unbounded diameter. We refer the reader to [2, 3] for an extensive discussion of distance-regular graphs, and to [1, 5] for more information on the twisted Grassmann graphs.

Jungnickel and the second author [9] constructed a family of designs which have the same parameters as $\mathrm{PG}_e(2e,q)$, and showed that these designs give the first infinite family of counterexamples to Hamada's conjecture [6, 7]. The purpose of this note is to show that the twisted Grassmann graph is the block graph of the design constructed in [9], just as the Grassmann graph is the block graph of the design $\mathrm{PG}_e(2e,q)$.

2. Statements of the result

Let H be a fixed hyperplane of V. The twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ (see [4]) has a set of vertices $\mathcal{A} \cup \mathcal{B}$, where

$$\mathcal{A} = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H\},$$
$$\mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix}.$$

Date: July 4, 2009.

Key words and phrases. distance-regular graph, Grassmann graph, projective geometry, design.

The adjacency is defined as follows:

$$W_1 \sim W_2 \iff \begin{cases} \dim W_1 \cap W_2 = e & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{A}, \\ W_1 \supset W_2 & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{B}, \\ \dim W_1 \cap W_2 = e-2 & \text{if } W_1 \in \mathcal{B},\ W_2 \in \mathcal{B}. \end{cases}$$

Let σ be a polarity of H. That is, σ is an inclusion-reversing permutation of the set of subspaces of H, such that σ^2 is the identity. Then $\sigma(W_1) \cap \sigma(W_2) = \sigma(W_1 + W_2)$ holds for any subspaces W_1, W_2 of H. We refer the reader to [8] for details on polarities.

The pseudo-geometric design constructed by Jungnickel and Tonchev [9] has $[V]$ as the set of points, and $\mathcal{A}' \cup \mathcal{B}'$ as the set of blocks, where

$$\mathcal{A}' = \{[\sigma(W \cap H) \cup (W \setminus H)] \mid W \in \mathcal{A}\},$$
$$\mathcal{B}' = \{[W] \mid W \in \begin{bmatrix} H \\ e+1 \end{bmatrix}\}.$$

It is shown in [9] that the incidence structure $([V], \mathcal{A}' \cup \mathcal{B}')$ is a 2-(v,k,λ) design, where

$$v = \frac{q^{2e+1}-1}{q-1},\ k = \frac{q^{e+1}-1}{q-1},\ \lambda = \frac{(q^{2e-1}-1)\cdots(q^{e+1}-1)}{(q^{e-1}-1)\cdots(q-1)}.$$

Moreover, as shown in [9], the sizes of the intersections of pairs of blocks are

$$\frac{q^i-1}{q-1} \quad (i = 1, \dots, e),$$

which are exactly the same as those for the geometric design $\mathrm{PG}_e(2e,q)$. This leads us to define the block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$ in the same manner as in $\mathrm{PG}_e(2e,q)$, and it turns out that this block graph is isomorphic to the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$.

Theorem 1. *The twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ is isomorphic to the block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$, where two blocks are adjacent if and only if their intersection has size $(q^e-1)/(q-1)$.*

Proof. We define a mapping $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{A}' \cup \mathcal{B}'$ by

$$f(W) = \begin{cases} [\sigma(W \cap H) \cup (W \setminus H)] & \text{if } W \in \mathcal{A}, \\ [\sigma(W)] & \text{if } W \in \mathcal{B}. \end{cases}$$

It suffices to show

$$W_1 \sim W_2 \iff |f(W_1) \cap f(W_2)| = \frac{q^e-1}{q-1}. \tag{1}$$

If W_1, W_2 are subspaces of V, then

$$\begin{aligned} &\dim \sigma(W_1 \cap H) \cap \sigma(W_2 \cap H) \\ &= \dim \sigma(W_1 \cap H + W_2 \cap H) \\ &= 2e - \dim W_1 \cap H - \dim W_2 \cap H + \dim W_1 \cap W_2 \cap H \end{aligned}$$

$$= \begin{cases} \dim W_1 \cap W_2 & \text{if } W_1, W_2 \in \mathcal{A},\ W_1 \cap W_2 \subset H \\ \dim W_1 \cap W_2 - 1 & \text{if } W_1, W_2 \in \mathcal{A},\ W_1 \cap W_2 \not\subset H \\ \dim W_1 \cap W_2 + 1 & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{B}, \\ \dim W_1 \cap W_2 + 2 & \text{if } W_1, W_2 \in \mathcal{B} \end{cases}$$

Thus, if $W_1, W_2 \in \mathcal{A}$, then

$$\begin{aligned} &|f(W_1) \cap f(W_2)| \\ &= |[\sigma(W_1 \cap H) \cap \sigma(W_2 \cap H)]| + |[W_1 \cap W_2 \setminus H]| \\ &= \begin{cases} \dfrac{q^{\dim W_1 \cap W_2} - 1}{q-1} & \text{if } W_1 \cap W_2 \subset H, \\ \dfrac{q^{\dim W_1 \cap W_2 - 1} - 1}{q-1} + \dfrac{q^{\dim W_1 \cap W_2} - q^{\dim W_1 \cap W_2 - 1}}{q-1} & \text{otherwise} \end{cases} \\ &= \frac{q^{\dim W_1 \cap W_2} - 1}{q-1}, \end{aligned}$$

and hence (1) holds.

Similarly, if $W_1 \in \mathcal{A}$, $W_2 \in \mathcal{B}$, then

$$|f(W_1) \cap f(W_2)| = \frac{q^{\dim W_1 \cap W_2 + 1} - 1}{q-1},$$

and hence

$$\begin{aligned} |f(W_1) \cap f(W_2)| = \frac{q^e - 1}{q-1} &\iff \dim W_1 \cap W_2 = \dim W_2 \\ &\iff W_1 \sim W_2. \end{aligned}$$

Finally, if $W_1, W_2 \in \mathcal{B}$, then

$$|f(W_1) \cap f(W_2)| = \frac{q^{\dim W_1 \cap W_2 + 2} - 1}{q-1}.$$

and hence (1) holds. □

3. The automorphism group

Let $\Gamma\mathrm{L}(V)_H$ denote the stabilizer of the hyperplane H in the general semilinear group $\Gamma\mathrm{L}(V)$ on V. For each $\phi \in \Gamma\mathrm{L}(V)_H$, we define a permutation ϕ' on $[V]$ by

$$\phi'(\langle x \rangle) = \begin{cases} \sigma\phi\sigma(\langle x \rangle) & \text{if } \langle x \rangle \in [H], \\ \phi(\langle x \rangle) & \text{otherwise}, \end{cases} \tag{2}$$

where $\langle x \rangle$ denotes the 1-dimensional subspace spanned by a nonzero element $x \in V$. It is straightforward to verify that ϕ' is an automorphism of the design $([V], \mathcal{A}' \cup \mathcal{B}')$. Indeed, suppose $W \in \begin{bmatrix} V \\ e+1 \end{bmatrix}$, $W \not\subset H$. Then

$$\begin{aligned} &\phi'([\sigma(W \cap H) \cup (W \setminus H)]) \\ &= \{\phi'(\langle x \rangle) \mid \langle x \rangle \in [\sigma(W \cap H) \cup (W \setminus H)]\} \\ &= \{\sigma\phi\sigma(\langle x \rangle) \mid \langle x \rangle \in [\sigma(W \cap H)] \cup \{\phi(\langle x \rangle) \mid \langle x \rangle \in [W \setminus H]\} \\ &= \{\langle x \rangle \mid \sigma\phi(W \cap H) \supset \langle x \rangle \in [H]\} \cup [\phi(W) \setminus H] \end{aligned}$$

$$= [\sigma(\phi(W) \cap H) \cup \phi(W) \setminus H)]$$
$$\in \mathcal{A}'.$$

Next suppose $W \in \begin{bmatrix} H \\ e+1 \end{bmatrix}$. Then

$$\begin{aligned}
\phi'([W]) &= \{\sigma\phi\sigma(\langle x \rangle) \mid \langle x \rangle \in [W]\} \\
&= \{\langle x \rangle \mid \sigma\phi\sigma(W) \subset \langle x \rangle \in [H]\} \\
&= [\sigma\phi\sigma(W)] \\
&\in \mathcal{B}'.
\end{aligned}$$

Therefore, ϕ' is an automorphism of the design $([V], \mathcal{A}' \cup \mathcal{B}')$.

Theorem 2. *Every automorphism of the design* $([V], \mathcal{A}' \cup \mathcal{B}')$ *is of the form* (2)*, and the full automorphism group of the design* $([V], \mathcal{A}' \cup \mathcal{B}')$ *is isomorphic to* $P\Gamma\mathrm{L}(V)_H$.

Proof. Let α be an automorphism the design $([V], \mathcal{A}' \cup \mathcal{B}')$. By abuse of notation, denote by the same α the permutation of $\mathcal{A}' \cup \mathcal{B}'$ induced by α. Then Theorem 1 implies that $f^{-1}\alpha f$ is an automorphism of the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$. Since the automorphism group of $\tilde{J}_q(2e+1, e)$ is $P\Gamma\mathrm{L}(V)_H$ by [5], there exists an element $\phi \in \Gamma\mathrm{L}(V)_H$ such that $f^{-1}\alpha f(W) = f\phi(W)$ for all $W \in \mathcal{A} \cup \mathcal{B}$. Then it is easy to verify that $\alpha(B) = \phi'(B)$ for all $B \in \mathcal{A}' \cup \mathcal{B}'$. Indeed, suppose $W \in \mathcal{A}$, so that $[\sigma(W \cap H) \cup (W \setminus H)] \in \mathcal{A}'$. Then

$$\begin{aligned}
\alpha([\sigma(W \cap H) \cup (W \setminus H)]) &= \alpha f(W) \\
&= f\phi(W) \\
&= [\sigma(\phi(W) \cap H) \cup (\phi(W) \setminus H)] \\
&= [\sigma\phi\sigma(\sigma(W \cap H)) \cup \phi(W \setminus H)] \\
&= \phi'([\sigma(W \cap H) \cup (W \setminus H)]).
\end{aligned}$$

Next suppose $W \in \mathcal{B}$, so that $[\sigma(W)] \in \mathcal{B}'$. Then

$$\begin{aligned}
\alpha([\sigma(W)]) &= \alpha f(W) \\
&= f\phi(W) \\
&= [\sigma\phi(W)] \\
&= [\sigma\phi\sigma(\sigma(W))] \\
&= \phi'([\sigma(W)]).
\end{aligned}$$

Therefore $\alpha(B) = \phi'(B)$ for all $B \in \mathcal{A}' \cup \mathcal{B}'$.

Since the action of an automorphism of a 2-design on blocks uniquely determines the action on points if the design has no repeated blocks, we obtain the desired result. □

Acknowledgments

The second author thanks the Graduate School of Information Sciences at Tohoku University, Sendai, Japan, for the hospitality and support during his visit in June 2009 as a Fulbright Senior Specialist, Project #3388. This author acknowledges also partial support by NSA Grant H98230-08-1-0065.

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